Topology from Differentiable Viewpoint

J. Milnor (Note by Conan Leung)

KEY: 1 dim mfd M : S^1 or $[0,1]$

$\Rightarrow \#(\partial M) = 0$ or $2 \equiv 2 \pmod 2$

or "$\#$" $(\partial M) = 0$ or $1 + (-1) = 0$

(All given maps are assumed smooth)

§ Sard's theorem

<u>Theorem</u>: $f : M^m \longrightarrow N^n$ w/ $m \geq n$

$\Rightarrow$ alm. all $y \in N$, y is regular value

[i.e. $\forall$ $f(x) = y$, $df(x) : T_x M \xrightarrow{\text{onto}} T_{f(x)} N$

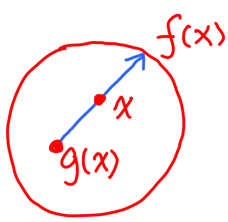
In particular, $f^{-1}(y)$ mfd. of dim $m-n$, or empty

- Cor: $\nexists$ $f : M \longrightarrow \partial M$ w/ $f|_{\partial M} = 1$

  [Pf. of cor: For regular $y \in \partial M$
  $f^{-1}(y)$ 1 dim. mfd. w/ $\partial(f^{-1}(y)) = \{y\}$ $(\rightarrow\!\!\!\!\times)$

- Cor: (Brouwer fixed point theorem)

  $g : D^n \longrightarrow D^n \Longrightarrow \exists\, x$ s.t. $g(x) = x$

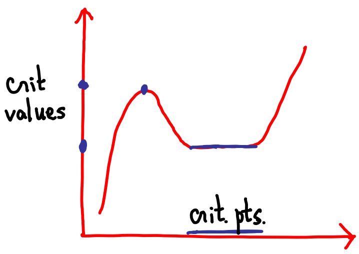


[Pf: If NOT $\Rightarrow \exists\, f$ as above.

Sand's theorem. $f: \mathbb{R}^m \to \mathbb{R}^p$ (or $M^m \xrightarrow{f} N^p$)

Crit. set $C := \{x \mid \operatorname{rk}(df(x)) \lneq p\}$

$\Rightarrow$ $f(C)$ has Lebesgue measure 0.



$\{$ critical points $\} \subseteq$ domain can be big

$\{$ critical values $\} \subseteq$ target is small

Pf: Use induction on p and $\forall$ measurable $A \subset \mathbb{R}^1 \times \mathbb{R}^{p-1}$

$$\operatorname{mea}(A \cap (t \times \mathbb{R}^{p-1})) = 0 \ \forall t \xRightarrow{\text{Fubini}} \operatorname{mea}(A) = 0$$

1° $\forall \bar{x}$ w/ $0 \neq df(\bar{x}) = \begin{array}{c} \left(\begin{array}{c|c} \neq 0 & * \\ \hline * & * \end{array} \right) \Big\} n \\ \underbrace{\qquad\qquad}_{p} \end{array}$

(say $0 \neq \frac{\partial f_1}{\partial x_1}(\bar{x})$)

Change coord. on $\mathbb{R}^n$ s.t. (locally)

$$\begin{array}{ccc} f_{new}: \mathbb{R}^1 \times \mathbb{R}^{n-1} & \longrightarrow & \mathbb{R} \times \mathbb{R}^{p-1} \\ \cup & & \cup \\ t \times \mathbb{R}^{n-1} & \xrightarrow[f^t_{new}]{} & t \times \mathbb{R}^{p-1} \end{array} \qquad \forall t$$

(Indeed $f_{new} = f \circ h$ w/ $h(x_1, x_2, \dots) = (f_1(x), x_2, \dots) : \mathbb{R}^n \circlearrowleft$)

$$d f_{new} = \left(\begin{array}{c|c} 1 & 0 \\ \hline * & * \end{array} \right)$$

So, crit. pt. of f^t_{new} $\longleftrightarrow$ crit. pt. of f_{new} at $(t, \cdots)$
$\longleftrightarrow$ crit pt. of f at $\cdots\cdots$

Induct^n on p + Fubini $\Rightarrow$ $\operatorname{mea}(C \setminus \{df = 0\}) = 0$.

2° Similarly, $\text{mea}(\{df=0\}\setminus\{df=D^2f=0\})=0$ etc.

3° Remain to show $\text{mea}(\{D^{\leq k}f=0\})=0$

if k large enough $(k>\frac{n}{p}-1)$.

$$(D^{\leq k}f)(x)=0 \xRightarrow[\text{Taylor}]{} f(x+h)=f(x)+R(x,h)$$

w/ $|R|\leq C|h|^{k+1}$

Divide $I^n=[0,1]^n$ into N^n cubes of size $[0,\frac{1}{N}]^n$

$$\left.\begin{array}{l}(D^{\leq k}f)(x)=0\\ x\in[0,\frac{1}{N}]^n \text{ (translated)}\end{array}\right\}\xRightarrow{\text{Taylor}} f\Big(\underbrace{[0,\tfrac{1}{N}]^n}_{\text{(translated)}}\Big)\subseteq\Big[0,\frac{a}{N^{k+1}}\Big]^p \text{ (translated)}.$$

$$\Rightarrow \text{mea}(\{D^{\leq k}f=0\}\cap I^n)\leq N^n\cdot\Big(\frac{a}{N^{k+1}}\Big)^p\xrightarrow[N\to\infty]{}0$$

QED.

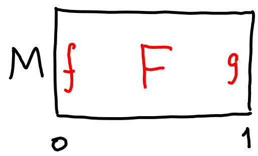
§ mod 2 degree

Def. $f, g : M \longrightarrow N$ (smooth) homotopic

if $\exists\ F : M \times [0,1] \longrightarrow N$ (smooth)

s.t. $F|_{M\times 0} = f$

$F|_{M \times 1} = g$



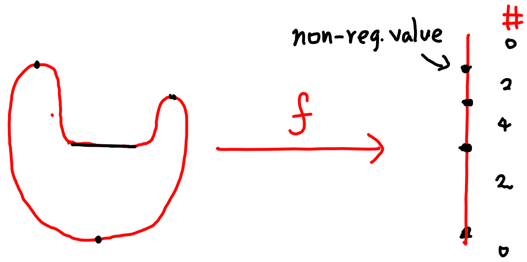
- $y \in N$ regular value

$\Rightarrow f^{-1}(y) \subset M$ smooth submfd

Assume $\begin{cases} \dim M = \dim N & \Rightarrow f^{-1}(y) \text{ 0-dim} \\ M \text{ cpt.} & \Rightarrow \# f^{-1}(y) < \infty \end{cases}$

- $y \longmapsto \# f^{-1}(y)$

locally const. fu. $N^{reg} \longrightarrow \mathbb{N}$



<u>Theorem</u>: $y, z \in N^{reg}$ (N connected)

$\Longrightarrow \# f^{-1}(y) \equiv \# f^{-1}(z) \pmod 2$

[Key : classificat^{n} of 1 dim. mfd : $\bigcirc$ or $\vdash\!\!\dashv$ $\Rightarrow$ # bdy pt $= 0$ or $2 \in 2\mathbb{Z}$]

Cor : $1_M \not\simeq \text{Const} : \underset{\text{closed}}{M} \longrightarrow M$

(closed mfd. = compact w/o boundary)

Lemma 1: $f \overset{F}{\sim} g : M^n \longrightarrow N^n$, M closed

$y \in N$ regular for both f & g

$$\Rightarrow \# f^{-1}(y) \equiv \# g^{-1}(y) \pmod 2$$

[Pf: Perturb $y \rightsquigarrow y' \in N$ regular for F

$\Rightarrow$ $F^{-1}(y')$ 1-mfd (i.e. union of S^1 & $[0,1]$)

$$\Rightarrow \# \underbrace{\partial F^{-1}(y')}_{f^{-1}(y')\times 0 \,\cup\, g^{-1}(y')\times 1} \in 2\mathbb{Z}$$

$$\Rightarrow \# f^{-1}(y') \equiv \# g^{-1}(y') \pmod 2$$

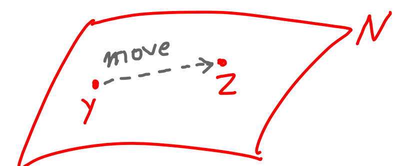
$\quad\;\, \| \qquad\qquad\qquad \|$

$\# f^{-1}(y) \qquad \# g^{-1}(y)$ $\quad$ ($\because$ $\#$: loc. const. fu.)

Lemma 2. $y, z \in N$ connected

$\Rightarrow \exists\, h : N \longrightarrow N$ diffeo.

$h(y) = z$



($\sim$ water "flow")

and $h \overset{F}{\sim} 1_N$ isotopic ($\overset{\Delta}{\Leftrightarrow}$ $F|_{X\times t}$ diffeo. $\forall t$)

[Proof of theorem:

$$M \xrightarrow{f} N \xrightarrow[\text{(isotopy)}]{\text{(lemma 2)}\ \exists h} N$$

$y \longmapsto z$ $\quad$ z reg. for f & $h\circ f$

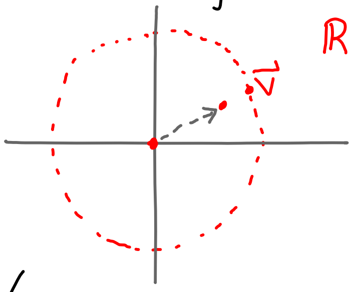
$$h \sim 1_N \Rightarrow h\circ f \sim 1\circ f = f$$

$$\overset{\text{lemma 1}}{\Longrightarrow} \# f^{-1}(\underbrace{h^{-1}(z)}_{y}) \equiv \# f^{-1}(z) \pmod 2$$

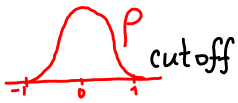
QED.

Proof of lemma 2 : Local issue, say moving $o \in \mathbb{R}^n$ to nearby pt. w/o disturbing $\mathbb{R}^n \setminus D$

$$\frac{d\vec{x}}{dt} = \vec{v} \cdot \rho(|\vec{x}|)$$

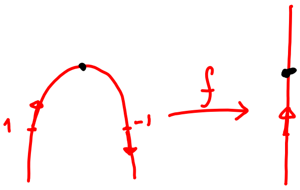


$|\vec{v}| = 1$,



ODE $\xrightarrow{\exists \text{ sol}^n}$ flow on $\mathbb{R}^n$ ✓

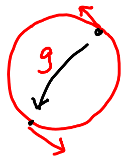
Question: $\# f^{-1}(y) = \# f^{-1}(z)$ w/o mod 2 ?



#	deg
0	0
2	$1+(-1)=0$

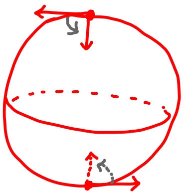
$\Rightarrow$ need a sign!

Eg. $g : S^1 \longrightarrow S^1$, $g(\vec{x}) = -\vec{x}$
antipodal $\qquad |\vec{x}| = 1$



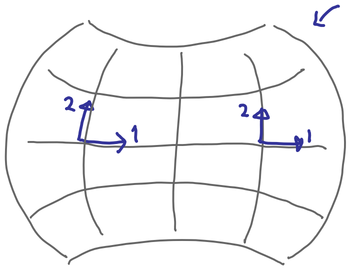
" $\deg g = +1$ "

Eg $g : S^n \longrightarrow S^n$
antipodal

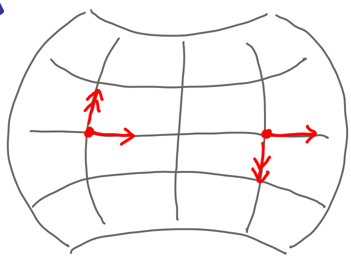


↙ world map ↘

orientation



g



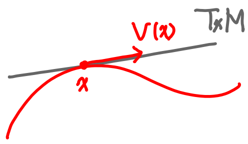
1 direction matches
other directions reverse

" $\deg g = (-1)^{n-1}$ "

- $g \sim 1_{S^n} \Longrightarrow n \in 2\mathbb{Z}+1$
  $\left(\overset{\text{true}}{\Longleftarrow}\right)$

Cor. S^n admits nonvanishing vector field (v.f)

$\iff n \in 2\mathbb{Z}+1$

Recall  $M \subset \mathbb{R}^N$

v.f. $\leftrightarrow$ $v : M \longrightarrow \mathbb{R}^N$

$v(x) \in T_xM \quad \forall\, x$

When $M = S^n = \{x : |x|^2 = x \cdot x = 1\} \subset \mathbb{R}^{n+1}$

$T_xS^n = \{v : v \cdot x = 0\} \subset \mathbb{R}^{n+1}$

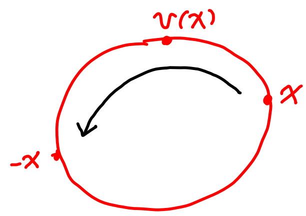
If v nonvanishing v.f. on S^n

$\Rightarrow v(x) \cdot x = 0$ & $v(x) \neq 0 \quad \forall\, x \in S^n$

$\Rightarrow V := \dfrac{v}{|v|} : S^n \longrightarrow S^n$

$\overset{\forall \theta}{\Longrightarrow} f_\theta := \cos\theta + V \sin\theta : S^n \longrightarrow S^n$

$(\because \sin^2\theta + \cos^2\theta = 1)$

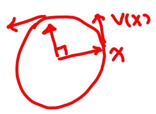


$\rightsquigarrow 1_{S^n} = f_0 \sim f_\pi = \overset{\text{antipodal}}{g}$

$\Rightarrow n \in 2\mathbb{Z}+1$ (by deg reason.)

($v(x)$ picks a route, among S^{n-1} choices, to move from x to $-x$)

[$\Leftarrow$] $n \in 2\mathbb{Z}+1$



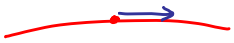
$v(x_1, x_2, \ldots\ldots) = (x_2, -x_1, \ldots\ldots)$ is an example.

Fact (Hopf) $f, g : \underset{\text{ori}}{M^n} \longrightarrow S^n$

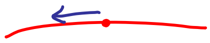
$f \sim g \iff \deg f = \deg g$

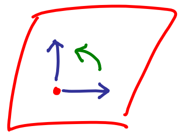
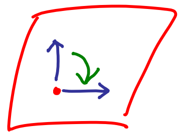
i.e. $\deg : [M^n, S^n] \longrightarrow \mathbb{Z}$ is bijection.

§ Orientation.

1D 

or



2D  or 

Linear algebra.

- $GL(n,\mathbb{R}) \xrightarrow{\det} GL(1,\mathbb{R}) = \mathbb{R}^\times$

  2 connected components

  In fact, $GL(n,\mathbb{R})$ has 2 —"— —"—
  according to $\det A > 0$ or $\det A < 0$.

- $V\ (\simeq \mathbb{R}^n) \Longrightarrow \det V \triangleq \Lambda^n V\ (\simeq \mathbb{R})$

  orientation $\longleftrightarrow$ $\nu \in \det V \setminus 0\ (\simeq \mathbb{R}^\times)$
  up to positive scaling.

  $\longleftrightarrow$ ordered basis $b_1, \dots, b_n$ of V
  up to $(b_1, \dots) \sim (b_1', \dots)$
  $b_i' = \sum_j a_i^j b_j$ s.t. $\det(a_i^j) > 0$

- Orientation for (connected) mfd. M^n

  $\Longleftrightarrow$ $\forall\, x \in M$, choose an ori. on $T_x M$

  (depending on $x \in M$ continuously)

  $\Longleftrightarrow$ $\nu \in \Gamma(M^n, \Lambda^n T^* M) =: \Omega^n(M)$
  $\nu(x) \neq 0 \quad \forall\, x$

- $\Lambda^n T^* M^n$ trivial $\mathbb{R}$-bundle
  $\Longleftrightarrow$ $\exists$ orientation
  $\Longrightarrow$ exactly 2 possible orientations

§ Degree

$$f : M^n \longrightarrow N^n \qquad f(x) = y$$

$$\rightsquigarrow \quad df(x) : T_x M \longrightarrow T_{f(x)} N \quad \cong \text{ if } y \text{ regular}$$

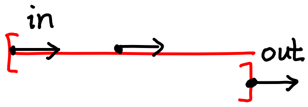
$$\det(-\text{"}-) : \underbrace{\Lambda^n T_x M}_{\mathbb{R}} \xrightarrow{\cong} \underbrace{\Lambda^n T_{f(x)} N}_{\mathbb{R}} \leftarrow \left(\begin{array}{l}\text{via orientations}\\ \text{on } M \text{ and } N.\end{array}\right)$$

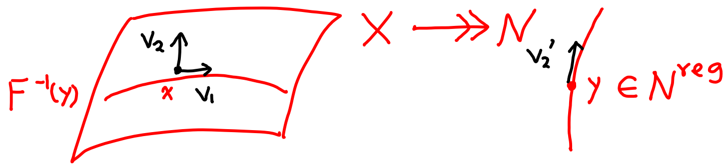
$$\rightsquigarrow \quad \pm 1 =: \operatorname{sign}(df(x))$$

Def: $\deg(f, -) : N^{reg} \longrightarrow \mathbb{Z}$

$$\deg(f, y) \triangleq \sum_{f(x) = y} \operatorname{sign}(df(x))$$

<u>Claim:</u> Indep. of $y \in N^{reg.}$

- 

- 

$$0 \longrightarrow T_x(F^{-1}(y)) \longrightarrow T_x X \longrightarrow T_y N \longrightarrow 0$$

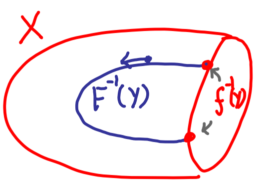
$$v_1 \;=\!=\!=\; v_1$$

$$v_2 \longmapsto v_2'$$

$$\mathrm{Ori}(X) \qquad \mathrm{Ori}(N)$$

$$\mathrm{Ori}(F^{-1}(y)) \Longleftarrow$$

- Lemma:
$$\begin{array}{c} \partial X \xrightarrow{\;f\;} N^n \ni y \\ \cap \quad \circlearrowleft \nearrow_{F} \\ X^{n+1} \end{array} \quad \underset{(\text{say } f \pitchfork F)}{\text{regular}} \Longrightarrow \deg(f, y) = 0$$



[Pf: Ori. of X + N $\Rightarrow$ Ori. for $F^{-1}(y)$ ← 1D mfd.

in vs out $\Rightarrow$ $\deg(f^{-1}(y)) = 0$

- $f \overset{F}{\sim} g : M^n \longrightarrow N^n \ni y \xRightarrow{\text{lemma.}} \deg(f, y) = \deg(g, y)$

- $f : M^n \longrightarrow N^n \ni y, z$ $\Rightarrow$ $\deg(f, y) = \deg(f, z)$
  isotopy y to z inside N i.e. claim. ✓

Theorem. $f : M^n \longrightarrow N^n$

(1) $\deg f \in \mathbb{Z}$ is well-def$^{\underline{d}}$.

(2) $f \sim g \Rightarrow \deg f = \deg g$

§ Index of tangent vector fields.

$v \in \Gamma(M, T_M)$ is called (tangent) vector field

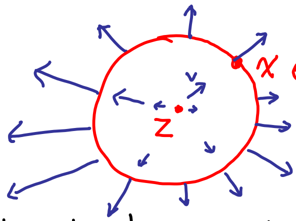
- $z \in M$ isolated zero of v

  $\rightsquigarrow$ $\operatorname{index}(v, z) \in \mathbb{Z}$

Theorem (Poincaré - Hopf theorem)

M compact $\Rightarrow \sum_z \operatorname{index}(v, z)$ indep. of v

(in fact, $= \Sigma(-1)^k \dim H_k(M, \mathbb{R})$).

· Defining index : 

$x \in \partial(B_\varepsilon(z)) = S^{n-1}$

Locally (up to perturbation / homotopy),

$z \in B_\varepsilon(z) \subset \mathbb{R}^n$ (a coord. chart)

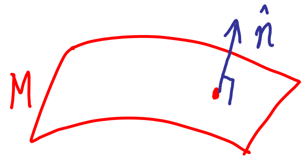
$$\underbrace{x \in \partial B_\varepsilon(z)}_{S^{n-1}_\varepsilon} \longmapsto 0 \neq v(x) \in T_x\mathbb{R}^n \simeq \mathbb{R}^n$$

($\because z$ isolated 0)

$$\longmapsto \frac{v(x)}{|v(x)|} \in S^{n-1}$$

$$\text{index}(v, z) \triangleq \deg\left(S^{n-1}_\varepsilon \xrightarrow{v/|v|} S^{n-1} \right)$$

(well-def^{d} $\because$ deg is inv. under homotopy).

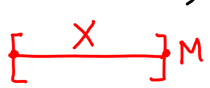
- $M^{n-1} \subset \mathbb{R}^n$ oriented hypersurface 

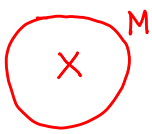
$\quad \rotatebox{90}{$\in$}$

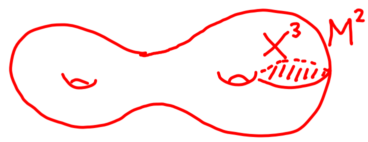
$\quad x \rightsquigarrow$ oriented unit normal $\hat{n}(x)$

i.e. Gauss map
$$G : M^{n-1} \longrightarrow S^{n-1}$$
$$x \longmapsto \hat{n}(x)$$

- Suppose $M = \partial X$ $\exists$ compact domain $X^n \subset \mathbb{R}^n$

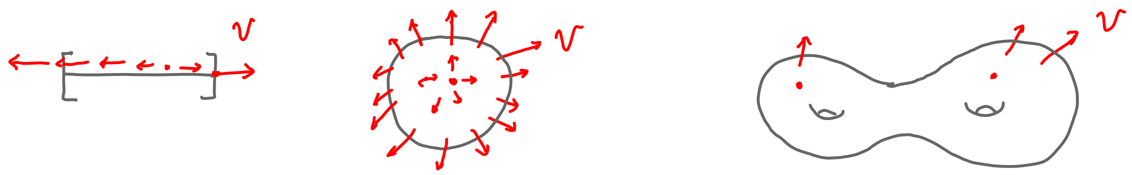
eg. $n=1$ 

, $n=2$ 

, $n=3$ 

Given vector field v on X s.t.

(i) w/ isolated zero, (ii) pointing outward along $\partial X = M$



On $X \setminus \bigcup_{v(z)=0} B_\varepsilon(z)$, v has no zero

$$\rightsquigarrow \left(\text{———} '' \text{———} \right) \xrightarrow{v/|v|} S^{n-1}$$

$$\Rightarrow \underbrace{\partial\left(\text{———} '' \text{———} \right)}_{M \,\cup\, \bigcup_{v(z)=0} \partial B_\varepsilon(z)} \xrightarrow{v/|v|} S^{n-1} : \deg = 0$$

$$\Rightarrow \deg G - \sum_{v(z)=0} \operatorname{index}(v, z) = 0$$

$\left(G \sim \frac{v}{|v|} \text{ on } \partial X \quad \because v \text{ pt. out.} \right)$

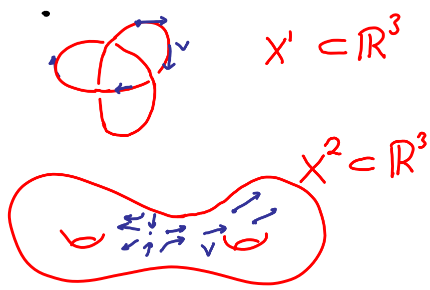
In particular, total index is indep. of v.

Above: v.f. on $X^n \underset{\text{domain}}{\subset} \mathbb{R}^n$ (w/ $\partial X \neq \emptyset$)

How about X^n w/ $\partial X = \emptyset$?

Say $X^n \subset \mathbb{R}^N$

v : v.f. on X



Assume non-degeneracy,

i.e. $\forall\ v(z) = 0$

$dv(z) : T_z X \longrightarrow T_z X$ non-singular

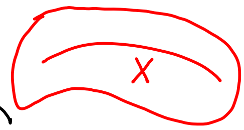
(in particular, $\operatorname{index}(v, z) = \pm 1$)

Theorem: v non-degen. v.f. on X^n $(\subset \mathbb{R}^N)$, then

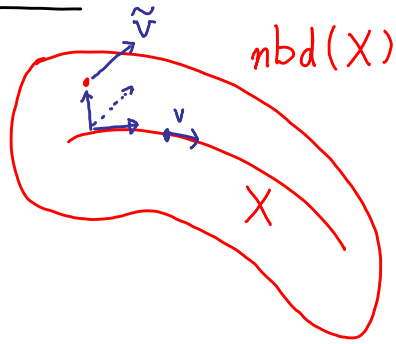
$$\sum_{v(z)=0} \operatorname{index}(v,z) = \deg G_{\partial(\operatorname{nbd}(X))}.$$

In particular, indep. of v.

$\left(\partial(\operatorname{nbd}(X)) \underset{\text{codim } 1}{\subset} \mathbb{R}^N \right.$



$\left. \right)$



$$\operatorname{nbd}(X) \simeq N_{X/\mathbb{R}^N} \quad r\downarrow \; \circlearrowleft \quad X$$

Extend v.f. v on X to v.f. $\tilde{v}$ on $\operatorname{nbd}(X)$

$$\tilde{v}(x) \triangleq x - r(x) + v(r(x)).$$

- $\tilde{v}$ pointing outward along $\partial(\operatorname{nbd}(X))$
- $\operatorname{Zero}(\tilde{v}) = \operatorname{Zero}(v) \ni z$

$$d\tilde{v}(z) = \begin{array}{c|c} \left(\begin{array}{c|c} dv(z) & 0 \\ \hline 0 & I \end{array}\right) & \begin{array}{l} T_zX \\ N_zX \end{array} \end{array}$$

$\Rightarrow$ same local index

QED.

In general, $\operatorname{Index}(v) = \chi(X)$ (Euler char.), for any vector field v on closed mfd. X w/ v having isolated zeros.

- $v(z) = 0 \Rightarrow \operatorname{index}(-v,z) = (-1)^{\dim X} \operatorname{index}(v,z)$

Therefore, $\chi(X^{2m+1}) = 0$.

- Fact: (Hopf) $\chi(X) = 0 \Rightarrow \exists$ nonvanishing v.f.

§ Framed cobordism

"$\deg(f: M^m \longrightarrow S^p)$" w/ $m \geq p$

IF $m = p$, (A) $\deg f \in \mathbb{Z}$

(B) $f \sim g \iff \deg f = \deg g$

Given $f : M^m \longrightarrow S^p$, $f^{-1}(y) \subset M^m$, $y \in S^p$ regular

- $f^{-1}(y)$: dim $m-p$ manifold
- $N_{f^{-1}(y)/M} \left(\overset{\sigma}{=} f^*(\underbrace{T_y S^p}_{\text{fix vector space}}) \right)$ trivial vector bundle.

called *framing* (i.e. trivializat$^{\text{n}}$ of normal bundle)

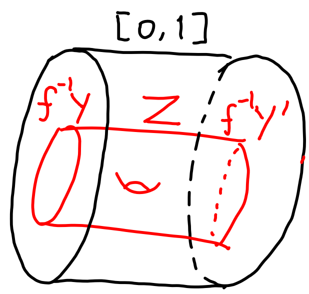
Lemma: $(f^{-1}(y), \sigma) \underset{\substack{\text{cobordant} \\ \text{inside } M}}{\overset{\text{framed}}{\sim}} (f^{-1}(y'), \sigma')$

i.e.

$f^{-1}(y) \times [0,\varepsilon) \cup f^{-1}(y') \times (1-\varepsilon, 1] \subset M \times [0,1]$

$\cap$

$\exists\ Z$



s.t. $\partial Z = f^{-1}(y) \times 0 \cup f^{-1}(y') \times 1$

& trivializations of normal bundles extend to Z.

Pf: (perturb a little) $Z = f^{-1}(y \bullet\!\!\frown\!\!\bullet y')$

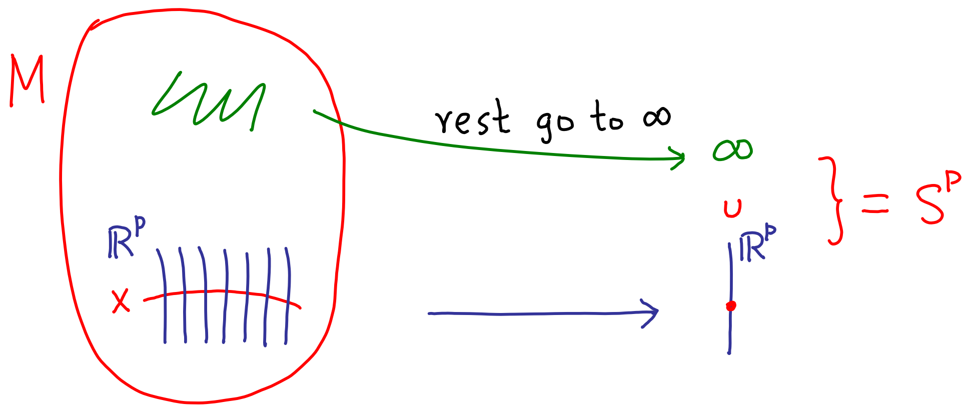
$\sigma_Z = f^{-1}\ \sigma_{y \in S^p} \sim \sigma_{y' \in S^p}$

Given $X^{m-p} \subset M^n$ w/ framing $N_{X/M} \overset{\sigma}{\simeq} X \times \mathbb{R}^p$

"Turbular nbd thm"

$$\begin{array}{ccccc} M & & & & S^p \\ \cup & & & & \cup \\ \mathrm{nbd}(X) & \simeq & X \times \mathbb{R}^p & \xrightarrow{\text{project}^n} & \mathbb{R}^p \\ \cup & & \cup & & \rotatebox{90}{$\in$} \\ X & = & X \times 0 & \longrightarrow & 0 \end{array}$$

$\sigma \quad \longleftrightarrow \quad$ std. basis on $\mathbb{R}^p$



$\rightsquigarrow \quad f : M^m \longrightarrow S^p \ni 0$ regular

s.t. $\quad X = f^{-1}(0) \quad \& \quad \sigma = f^{-1}(\mathrm{std})$

- That is $[M^m, S^p] \longrightarrow \left\{ \begin{array}{l} \text{codim } p \\ \text{fr. submfd. in } M \end{array} \right\} \Big/ \begin{array}{l} \text{framed} \\ \text{cobordant} \end{array}$

  is surjective.

<u>Theorem</u>: bijective.

Pf: Given $f, g : M \longrightarrow S^p$

If $(f^{-1}(y), f^*\sigma) \underset{\text{cobordant}}{\overset{\text{fr.}}{\sim}} (g^{-1}(y), g^*\sigma)$

$\overset{?}{\Rightarrow} f \sim g$

Claim: Can reduce to assuming (exercise).

$$(\underbrace{f^{-1}(y)}, f^*\sigma) = (g^{-1}(y), g^*\sigma)$$

Write $X := f^{-1}(y) = g^{-1}(y) \subset M$

$\overset{f^*\sigma = g^*\sigma}{\Longrightarrow}$ $f = g$ up to 1^{st} order nbd of $X \subset M$

$\overset{\text{deform a bit}}{\Longrightarrow}$ $f = g$ .... ∞^{th} order ....

i.e. honest nbd.

$$f, g : M \setminus X \longrightarrow S^p \setminus \{y\} \simeq \mathbb{R}^p$$

$\rightsquigarrow$ $f \overset{tf+(1-t)g}{\sim} g : M \setminus X \longrightarrow \mathbb{R}^p$ (using linear str. on $\mathbb{R}^p$)

Can extend to whole M

$(\because f = g \,|_{\text{nbd}(X \subset M)})$ QED.